# Real Homotopy and Hodge Theory in Kähler Geometry

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## Real Homotopy Theory of Smooth Manifolds

The real homotopy type of a smooth manifold $M$ is encoded in its de Rham algebra of differential forms $\Omega^*(M)$:

- By de Rham's Theorem, there is an isomorphism of rings
$$H^*_{\mathrm{dR}}(M) := H^*(\Omega^*(M)) \cong H^*_{sing}(M;\mathbb{R}).$$
- The algebra $\Omega^*(M)$ also encodes higher operations in cohomology, called *Massey products*.
- If $M$ is simply connected, then one can compute $\pi_*(M)\otimes\mathbb{R}$ out of the multiplicative structure of $\Omega^*(M)$:
$$\pi_n(M)\otimes\mathbb{R} \cong Hom(V^n,\mathbb{R}), \qquad (1)$$
where $\Lambda V \xrightarrow{\sim} \Omega^*(M)$ is a *cofibrant minimal replacement* of $\Omega^*(M)$.

However, $\Omega^*(M)$ is infinite-dimensional and the replacement is difficult to compute. A convenient situation is that of **formality**.

### Definition: (Formality)

A commutative dg-algebra $A$ is said to be **formal** if there is a string of quasi-isomorphisms of dg-algebras between $A$ and $H^*(A)$:
$$A \xleftarrow{\sim} \cdots \xrightarrow{\sim} H^*(A)$$
A manifold $M$ is said to be **formal** if $\Omega^*(M)$ is a formal real dg-algebra.

- For a formal dg-algebra $A$, one just needs to compute a cofibrant minimal replacement of $H^*(A)$.
- In particular, the real homotopy groups of a formal manifold $M$ can be computed just from the cohomology ring of $M$!

### Real versus Rational homotopy

- There is a quasi-isomorphism of commutative dg-algebras between $\Omega^*(M)$ and $\mathcal{A}_{pl}(M)\otimes_{\mathbb{Q}}\mathbb{R}$, where $\mathcal{A}_{pl}$ is Sullivan's functor of piece-wise linear forms on topological spaces.
- (Independence of base field) If $M$ is compact and $\Omega^*(M)$ is formal then $\mathcal{A}_{pl}(M)$ is a formal $\mathbb{Q}$-algebra.

### Examples:

1. Spheres are formal,
2. Symmetric spaces are formal,
3. Products of formal spaces are formal.

### Theorem: (Deligne-Griffiths-Morgan-Sullivan)

Every compact Kähler manifold is formal.

## Hodge Theory of Kahler Manifolds

- *In cohomology:*
The cohomology $H^n(M;\mathbb{R})$ of degree $n$ of a compact Kahler manifold comes equipped with a **pure Hodge structure** of weight $n$, that is:
$$H^n(M;\mathbb{R})\otimes\mathbb{C} = \bigoplus_{p+q=n} H^{p,q}, \quad \text{where } \overline{H^{p,q}} = H^{q,p}.$$
- *In homotopy:*
One might try to obtain new invariants by transferring this extra structure to the real homotopy groups. Doing that, one does **not** get a pure Hodge structure on homotopy. One gets instead a **mixed Hodge structure**.

### Definition: (Mixed Hodge Structure)

A **mixed Hodge structure** is a triple $(V,W,F)$ where

1. $V$ is a finite dimensional real vector space,
2. (Weight Filtration) $\cdots \subseteq W_{p-1} \subseteq W_p \subseteq \cdots \subseteq V$,
3. (Hodge Filtration) $\cdots \subseteq F^p \subseteq F^{p-1} \subseteq \cdots \subseteq V\otimes\mathbb{C}$.

For each $n\in\mathbb{Z}$, $F$ induces a pure Hodge structure of degree $n$ on
$$Gr^n_W(V) = \frac{W_n}{W_{n-1}}$$

### The case of $\mathbb{CP}^n$

Let us compute the real homotopy groups of $\mathbb{CP}^n$.

- Following (1), we build a minimal cofibrant replacement of $\Omega^*(\mathbb{CP}^n)$:
$$\begin{matrix} H^*(\mathbb{CP}^n;\mathbb{R}) \cong \mathbb{R}[x]/(x^n), \\ |x| = 2 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \Lambda(x,y) \xrightarrow{\sim} \Omega^*(\mathbb{CP}^n), \\ dx = 0, \quad |x| = 2 \\ dy = x^{n+1}, \quad |y| = 2n+1 \end{matrix}$$
- Lifting the pure Hodge structure on cohomology to $\Lambda(x,y)$, one has:
  - From compatibility of weight filtration with product and differential:
  $$\text{weight of } x = w(x) = 2 \implies w(y) = 2n+2$$
  - $y\in\pi_{2n+1}(\mathbb{CP}^n)\otimes\mathbb{R}$ has weight $2n+2 \leftarrow$ **mixed weights**.

## (Non-)formality of Hodge Structures

- **Question:** Since Kähler manifolds are formal, is the mixed Hodge structure on $\pi_n(M)\otimes\mathbb{R}$ a formal consequence of the pure Hodge structure on $H^*(M)$?
- **Answer:** No!


- Carlson, Clemens and Morgan (Ann. Sci. École Norm. Sup., 1981) give a family of geometric counterexamples.
- We wish to understand this phenomenon in a higher homotopy perspective!

## Work in Progress

### Mixed Hodge Diagrams

$$A = \Big(\ (A_{\mathbb{R}},W),\ (A_{\mathbb{C}},W,F),\ \phi : (A_{\mathbb{R}}\otimes\mathbb{C},W) \xrightarrow{\cong} (A_{\mathbb{C}},W)\ \Big)$$

$\downarrow$ Filtered $\mathbb{R}$-algebra; $\downarrow$ Bi-Filtered $\mathbb{C}$-algebra; $\downarrow$ Iso. of filtered $\mathbb{C}$-algebras

- The cohomology of a mixed Hodge diagram is a mixed Hodge diagram.

### Formality of Diagrams

We say that a diagram $A$ is formal if there is a string of diagram maps which are level-wise quasi-isomorphisms between $A$ and $H^*(A)$.

- Formality in this category implies level-wise formality, but not the other way around.
- Formality of diagrams implies that the Hodge structure on homotopy groups is a formal consequence of the cohomology!

### Obstructions to Formality

So we wish to study this stronger notion of formality.
To do so, we are generalizing to the setting of diagrams

- the obstructions to formality of Halperin and Stasheff (Adv. Math., 1979);
- the obstructions to formality of Kaledin (Moscow Math., 2005).

### Almost Complex Manifolds

The same techniques (homotopy theory of diagrams of algebras) will apply to study homotopy theory of complex and almost complex manifolds.
$$A = \Big(A_{\mathbb{R}},\ (A_{\mathbb{C}},F),\ \phi : A_{\mathbb{R}}\otimes\mathbb{C} \xrightarrow{\cong} A_{\mathbb{C}}\ \Big)$$



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